# Pre-class Warm-up!!!

Let f, g, h be three functions R -> R and consider their values at 0, 1 and 2. Which of the following are logically correct statements?

a. If f, g, h are linearly independent then so are the three vectors $\begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix}$ $\begin{bmatrix} g(0) \\ g(1) \\ g(2) \end{bmatrix}$ $\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}$

Correct Most

Incorrect ✓ some

b. If f, g, h are dependent, then so are the three vectors $\begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix}$ $\begin{bmatrix} g(0) \\ g(1) \\ g(2) \end{bmatrix}$ $\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}$

Correct ✓ More

Incorrect fewer

If $af + bg + ch = 0$ zero function

then $af(0) + bg(0) + ch(0) = 0$

etc

c. If the three vectors $\begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix}$ $\begin{bmatrix} g(0) \\ g(1) \\ g(2) \end{bmatrix}$ $\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}$ are linearly independent then so are f, g, h.

✓ Correct

Incorrect Most

d. If the three vectors $\begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix}$ $\begin{bmatrix} g(0) \\ g(1) \\ g(2) \end{bmatrix}$ $\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix}$ are linearly dependent then so are f, g, h.

Correct

✓ Incorrect

c is the contrapositive of b so c is correct.

d a

We used c. last time with $e^x$, $\sin x$, 1

so $a\begin{bmatrix} f(0) \\ f(1) \\ f(2) \end{bmatrix} + b\begin{bmatrix} g(0) \\ g(1) \\ g(2) \end{bmatrix} + c\begin{bmatrix} h(0) \\ h(1) \\ h(2) \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$

## Section 5.1: second order linear differential equations
## Section 5.2: higher order linear differential equations

These two sections do the same thing.

Vocabulary review:

- Linear, homogeneous differential equations
- Solution space, initial value problem
- Linearly independent solutions

New vocabulary

- superposition of solutions = the solutions to a homogeneous equation form a vector space.
- characteristic equation

We learn:

- The Wronskian
- Solutions of linear homogeneous d.e.'s form a vector space.
- How to use the characteristic equation to solve homogeneous equations with constant coefficients
- What to do about non-homogeneous equations: complementary functions.

Question 5 from Section 5.1 (like questions 3 and 10 on the HW).

Given two solutions
y_1 = e^x and y_2 = e^{2x}
of the equation y'' - 3y' + 2y = 0
find a particular solution with
y(0) = 1 and y'(0) = 0

homogeneous, linear, 2nd order

$y_1 = e^x$ and $y_2 = e^{2x}$ are solutions

Here, all functions $c_1 y_1 + c_2 y_2$ where $c_1, c_2$ are numbers are solutions. We find $c_1, c_2$.

Note: $(c_1 y_1 + c_2 y_2)' = c_1 e^x + 2c_2 e^{2x}$

$c_1 y_1 + c_2 y_2 = c_1 e^x + c_2 e^{2x}$

Put $x = 0$

$$c_1 + c_2 = 1$$
$$c_1 + 2c_2 = 0$$

$$\begin{bmatrix} 1 & 1 \\ 1 & 2 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$c_1 = \frac{\begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}} \qquad c_2 = \frac{\begin{vmatrix} 1 & 1 \\ 1 & 0 \end{vmatrix}}{\begin{vmatrix} 1 & 1 \\ 1 & 2 \end{vmatrix}}$$

$$c_1 = \frac{2}{1} = 2 \qquad c = \frac{-1}{1} = -1$$

The particular solution is

$$y = 2e^x - e^{2x}.$$

## Independence using the Wronskian

**Definition**
Suppose we are given n functions $y_1, \ldots, y_n$
Their Wronskian is the function

$$W(x) = \det \begin{bmatrix} y_1 & \cdots & y_n \\ y_1' & \cdots & y_n' \\ y_1'' & & y_n'' \\ \vdots & & \vdots \\ y_1^{(n-1)} & \cdots & y_n^{(n-1)} \end{bmatrix}$$

**Theorem** (easy and useful part of bigger theorem)

If $y_1, \ldots, y_n$ are linearly dependent functions then $W(x) \equiv 0$ (the zero function)

Proof. If $c_1 y_1 + \cdots + c_n y_n \equiv 0$ then
$c_1 y^{(r)} + \cdots + c_n y_n^{(r)} \equiv 0$ for every $r$.
The columns of the Wronskian matrix are dependent. Thus its det $\equiv 0$

Corollary. If $W(x) \not\equiv 0$ then $y_1, \ldots, y_n$ are independent.

5.1 question 26 (like question 25)
Show that $2\cos x + 3\sin x$, $3\cos x - 2\sin x$ are independent.

Solution.

$$W(x) = \begin{vmatrix} 2\cos x + 3\sin x & 3\cos x - 2\sin x \\ -2\sin x + 3\cos x & -3\sin x - 2\cos x \end{vmatrix}$$

$$= (2\cos x + 3\sin x)(-3\sin x - 2\cos x) - (-2\sin x + 3\cos x)(3\cos x - 2\sin x)$$

$$= (-9-4)\sin^2 x + (-4-9)\cos^2 x + (-12+12)\sin x \cos x$$

$$= (-13)(\sin^2 x + \cos^2 x) = -13 \neq 0$$

Thus the functions are independent.

Example done for section 4.7.
Are the functions e^x, sin x and 1 linearly independent?

Solution

$$W(x) = \begin{vmatrix} e^x & \sin x & 1 \\ e^x & \cos x & 0 \\ e^x & -\sin x & 0 \end{vmatrix}$$

$$= \begin{vmatrix} e^x & \cos x \\ e^x & -\sin x \end{vmatrix} = -e^x(\sin x + \cos x)$$

$$\neq 0$$

Thus the functions are independent.

# Pre-class Warm-up!!!

Which of the following statements did we prove last time?

Let y_1, … , y_n be functions of x and let W(x) be their Wronskian.

✓a. If y_1, … , y_n are dependent then W(x) is identically zero.

b. If W(x) is identically zero then y_1, … , y_n are dependent.

c. If y_1, … , y_n are independent then W(x) is not identically zero

✓d. If W(x) is not identically zero then y_1, … , y_n are independent

e. We didn't prove any of these last time.

Can we even remember what the Wronskian is?

f. Are we comfortable making an adjective into a noun?

## Theorems about the existence of solutions

The following theorem combines
from Section 5.1: Theorems 1, 4
from Section 5.2: Theorems 1, 4.

Theorem. The solutions to the nth order linear d.e.

$$y^{(n)} + p_1(x)y^{(n-1)} + \cdots + p_{n-1}(x)y' + p_n(x)y = 0$$

where p_1, … , p_n are continuous
form a vector space of dimension n.

Theorem 2. For each number a and for all numbers b_0, … , b_(n-1), there is a unique solution y with
y(a) = b_0, y′(a) = b_1, … , y^(n-1) (a) = b_(n-1)

Assuming theorem 2, prove theorem 1 that the space of solutions has dimension n.

Take an independent set of solutions $y_1, \dots, y_r$. To get a solution $y = c_1 y_1 + \cdots + c_r y_r$ satisfying the initial conditions we solve

$$\begin{bmatrix} y_1(a) & y_2(a) & \cdots & y_r(a) \\ y_1'(a) & y_2'(a) & \cdots & \vdots \\ \vdots & & & \\ y_1^{(n-1)}(a) & \cdots & & y_r^{(n-1)}(a) \end{bmatrix} \begin{bmatrix} c_1 \\ \vdots \\ c_r \end{bmatrix} = \begin{bmatrix} b_0 \\ \vdots \\ b_{n-1} \end{bmatrix}$$

If there is a solution, it is unique.
This implies $r \le n$. Now assume that $y_1, \dots, y_r$ is a maximal independent set.
It is a basis, and so $y_1, \dots, y_r$ span the solution space. There always is a solution to the matrix equation, so $r \geq n$.
We conclude $r = n$.

## Solving linear differential equations with constant coefficients: the characteristic equation

These look like $ay'' + by' + cy = 0$ where a, b, c are numbers.

Look for solutions of form $y = e^{rx}$

$y' = re^{rx}$ $\quad y'' = r^2 e^{rx}$

Substitute

$$ar^2e^{rx} + bre^{rx} + ce^{rx} = 0$$

$$e^{rx}(ar^2 + br + c) = 0.$$

We get solutions when

$$ar^2 + br + c = 0.$$

This is the characteristic equation.

Like Section 5.1 questions 33-42.
Find the general solution of the differential equation
$y'' - 2y' - 3y = 0$

Solution. The characteristic equation is

$r^2 - 2r - 3 = 0 \quad (r-3)(r+1) = 0$

$r = 3$ or $r = -1$

$y = e^{3x}$ $\quad y = e^{-x}$ are solutions.

The general solution is

$$y = c_1 e^{3x} + c_2 e^{-x}.$$

Like Section 5.1 questions 33-42.
Find the general solution of the differential equation
$y'' + 4y' + 4y = 0$

Solution: The characteristic equation is

$(r^2 + 4r + 4) = 0 \quad (r+2)^2 = 0$

$r = -2$, $y = e^{-2x}$ is a solution,

When there is a repeated root like this we get another solution

$y = xe^{-2x}$ so the general solution is $y = c_1 e^{-2x} + c_2 x e^{-2x}$

Check $y = xe^{-2x}$ is a solution

$y' = e^{-2x} - 2xe^{-2x}$

$y'' = -2e^{-2x} - 2e^{-2x} + 4xe^{-2x}$

$y'' + 4y' + 4y$

$= -4e^{-2x} + 4xe^{-2x} + 4e^{-2x} - 8xe^{-2x} + 4xe^{-2x}$

$= 0$

## A stronger theorem about the Wronskian

Theorem. Suppose the $n$ function $y_1, \dots y_n$ are solutions of a homogeneous $n$th order linear d.e. with continuous coefficients of $y, \dots, y^{(n-1)}$.

a. If they are dependent then their Wronskian is identically 0. We did this already.

b. If they are independent then their Wronskian is never 0.

For a proof of b. using 'Abel's formula' see 5.1 question 32 and 5.2 question 35.

## Particular and complementary solutions

Like 5.2 questions 21-24

24. A non-homogeneous d.e., a complementary solution y_c and a particular solution y_p are given.
Find a solution satisfying the initial conditions:

$$y'' - 2y' + 2y = 2x, \quad y(0) = 4, \quad y'(0) = 8$$

$$y_c = c_1 e^x \cos x + c_2 e^x \sin x$$

$$y_p = x + 1$$

Explanation:

A complementary solution is a solution to the corresponding homogeneous equation.
A particular solution is an actual solution.

The general solution has the form $y_c + y_p$ where $y_c$ is a general solution to the homogeneous eqn.

Solve $y(0) = y_c(0) + y_p(0)$

$= c_1 + 1 = 4$, $c_1 = 3$

$$y' = y_c' + y_p' = c_1 e^x \cos x - c_1 e^x \sin x + c_2 e^x \sin x + c_2 e^x \cos x + 1$$

$y'(0) = c_1 + c_2 + 1 = 8$ so $c_2 = 4$

The solution is

$$y = 3e^x \cos x + 4e^x \sin x + x + 1$$

## Non-homogeneous equations

Let $y^{(n)} + p_1(x) y^{(n-1)} + \cdots + p_n(x) y = f(x)$

be a non-homogeneous nth order linear d.e.
The associated homogeneous equation is the same, with 0 on the right instead of f(x).

Suppose the p_i are continuous.

Theorem 5.
Let y_p be a (particular) solution of the non-homogeneous equation, and let y_1, ... y_n be a basis for the solution space of the associated homogeneous equation.
Then every solution of the non-homogeneous equation can be written $y_p + y_c$

where y_c = c_1y_1 + ... + c_ny_n is a solution of the associated homogeneous equation.

Reason if $y$ is a solution then $y - y_p$ solves the homogeneous equation, so $y - y_p = y_c$ $= c_1 y_1 + \cdots + c_n y_n$.

and $y = y_p + y_c$.

What is the general solution to the equation $y'' - y = 0$ ?

a. $c_1 e^{2x} - c_2$

b. $c_1 x^2 + c_2 x$

c. $c_1 e^x + c_2 x e^x$

d. $c_1 e^x + c_2 e^{-x}$

e. None of the above.

where c_1, c_2 are constants.

$r^2 - 1 = 0 \qquad r = \pm 1$